MEASUREMENT OF THE TEMPERATURE OF TRAPPED ATOM POPULATIONS

Elementary derivation of the TOF signal profile ‡

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Introduction. John Essick and some of his students have for several years been working to perfect an "atom trap" based upon a design developed and described by Carl Wieman and associates.[1] The apparatus serves, according to Wieman, to trap a population of roughly 4×10^7 rubidium atoms. If the trapped population had a characteristic diameter of 0.1 mm then we would have

$$\text{atomic density } n \approx 4 \times 10^{13} \text{ cm}^{-3}$$

and

$$\text{typical interatomic separation } \ell \approx n^{-\frac{1}{3}} \approx 3 \times 10^{-5} \text{ cm}$$

We might plausibly anticipate that quantum effects may become pronounced if the individual atoms have

$$\text{de Broglie length } \lambda \approx \text{ interatomic separation } \ell$$

From

$$p = \sqrt{2mE} = \sqrt{2mkT} = h/\lambda$$

we expect such effects (**Bose-Einstein condensation!**) to become evident at temperatures lower than about

$$T_c = \frac{h^2}{2mk\ell^2} \approx 1.2 \times 10^{-10} \text{ K}$$

in ^{87}Rb vapor ($m = 1.45 \times 10^{-25}$ kg).

‡ Notes for a Reed College Physics Seminar presented 10 September 2003.

[1] C. Wieman, G. Flowers & S. Gilbert, "Inexpensive laser cooling & trapping experiment for undergraduate laboratories," AJP **63**, 317 (1995).

The confined atomic populations presented by atom traps are cool—by every familiar standard quite cold—but are roughly 100,000 times hotter than the temperatures required to achieve condensation. To reach those extreme low temperatures—to create "the coldest place in the universe," as Wieman likes to call the space within his apparatus—Wieman and Eric Cornell had to devise the cunning sequence of ultra-refrigeration strategies, and it was for the ultimate success of those that they shared (with Wolfgang Ketterle, of MIT) the Nobel Prize in 2001.[2]

During the question period following a seminar that Wieman presented at Reed College (15 November 2002) I asked how he *measured* the temperature of his condensates. His response: "We look [by a non-disruptive technique that he chose not to describe in detail[3]] to the [thermal component of] the velocity distribution."

That, I suppose, is the reason I developed special interest in the thesis work of Hannah Noble, who—building upon the work of Teresa Napili (1997), Ian Coddington (1998), Frederic Bahnson (1999), Sean Kellogg (2000) and Kalista Smith (2001)—had undertaken to measure the temperature of the rubidium vapor confined within Reed's MOT (Magneto-Optical Trap). And the reason that I invited myself to her Senior Oral... where I was able for the first time to skim her thesis.[4] Near the midpoint of those proceedings Hannah undertook to summarize her §2.3, which is concerned with the signal that is expected theoretically to result when a tightly confined population of atoms is dropped onto a "detection plane," and with the means by which temperature estimates can be extracted from such "Time-Of-Flight" (TOF) data. At the conclusion of Hanna's discussion I remarked that her argument—which I did not then realize had been adapted from a recent paper[5]—seemed to me to be unnecessarily complicated, that her final results could, I suspected, be obtained by a simpler and more transparent argument... which I proceeded to sketch. It was with Hannah's encouragement that I retired the next day to my office and wrote out the details, and it is some of those that I want to share with you today.[6]

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[2] See "BEC for everyone" on the JILA BEC homepage.

[3] The experimental details, I am assured by John Essick, are ingeniously intricate. And so must be the associated theory, for it is not immediately obvious what one means by the "velocity distribution" within a many-particle *quantum* system.

[4] Hannah D. Noble, "Time of flight: measuring the temperature of trapped atoms in the Reed MOT," Reed College thesis (2003).

[5] I. Yavin, M. Weel, A. Andreyuk & A. Kumarakrishnan, "A calculation of the time-of-flight distribution of trapped atoms," AJP **70**, 149 (2002). The paper appears to have been the work mainly of the first two authors, who at the time were undergraduates attached to the laboratory of the last-named author. Professor Kumarakrishnan (with whom I have been in correspondence) does trapped atom work at York University in Toronto, Canada.

[6] For omitted details see my "Measurement of trapped atom temperature: elementary theory of the TOF signal profile" (May 2003).

1. One-dimensional toy version of the argument. Fundamental to the kinetic theory of ideal gases is the statement (Maxwell 1860) that in a thermalized sample of N molecules one can expect to find that a number

$$dN = N \cdot \underbrace{\Big(\frac{1}{\alpha\sqrt{\pi}}\Big)^3 e^{-(v/\alpha)^2} 4\pi v^2}_{f(v)}\, dv \tag{1}$$

have speeds that lie within the neighborhood dv of v. Here $\alpha \equiv \sqrt{2kT/m}$ and m refers to the mass of the individual molecules. It follows that if such a "Maxwellian population" of runners were to race down a track of length s the number $F(t)$ of runners who will have reached the finish line by time $t > 0$ can be described

$$F(t) = N \int_{s/t}^{\infty} f(v)\, dv$$

and the *rate at which runners cross* the finish line becomes

$$\begin{aligned} R(t) \equiv \tfrac{d}{dt}F(t) &= N(s/t^2)f(s/t) \\ &= N\frac{4s^3}{\sqrt{\pi}\alpha^3 t^4} e^{-s^2/\alpha^2 t^2} \end{aligned} \tag{2}$$

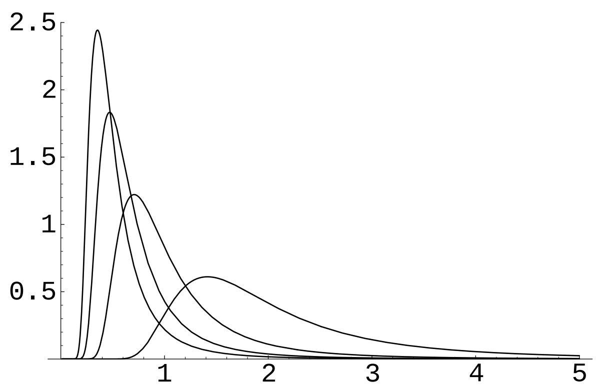

FIGURE 1: *Graphs of the activity $R(t)$ at the end of a Maxwellian race, based upon* (2)*. The race was of length $s = 1$ and the parameter α was assigned the values* 0.5 (*broad curve*), 1.0, 1.5, 2.0*. Larger α refers to hotter/faster runners, who finish more quickly.*

which has the form shown in Figure 1. One readily verifies that

$$\int_0^\infty R(t)\, dt = N \quad : \quad \text{all runners eventually finish}$$

and finds that $R(t)$ peaks at

$$t_{\max} = \tfrac{1}{\sqrt{2}}(s/\alpha) \tag{3}$$

It was, in point of historical fact, from TOF data $R(t)$ that Maxwell's theoretical result (1) received its first direct experimental support. Today, with (1) secured, we are in position to turn the procedure around: from the observed structure of $R(t)$ we extract a measured value of $t_{\max}$, which we use in (3)—written

$$T = \frac{ms^2}{4k(t_{\max})^2}$$

—to obtain an estimate of the temperature of the Maxwellian population of runners/molecules. It is a variant of that procedure, now standard to what we might call the "atom trap industry," that concerns us:

2. Dropped gas balls. The basic set-up is shown in Figure 2, and makes physical sense only if the blob is so cold that it remains reasonably compact for the

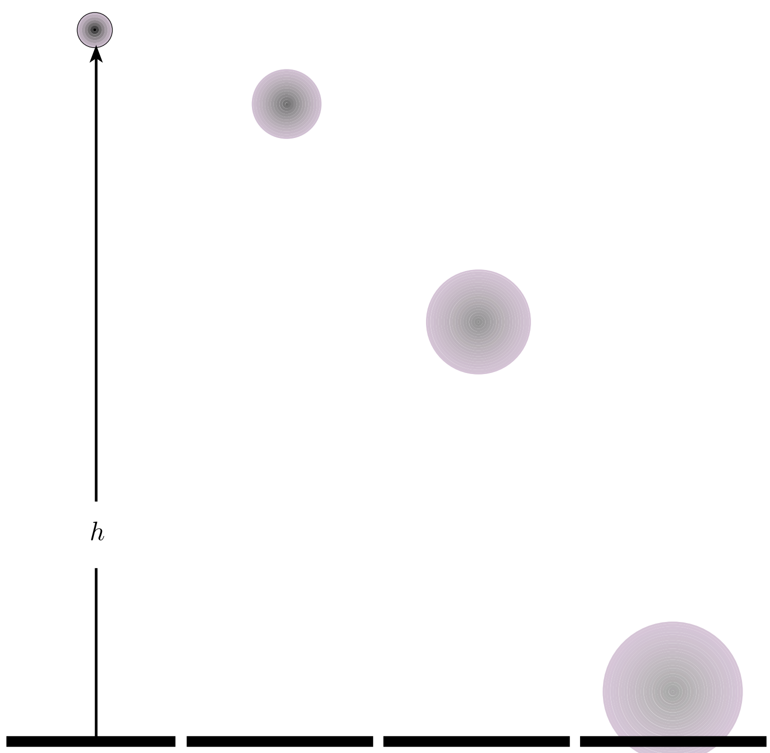

FIGURE 2: *Successive snapshots of a little blob of very cold vapor that has been dropped onto a sheet of laser light. Individual atoms flouresce and are counted as they pass through the sheet. Our assignment is deduce the theoretical profile of the signal $S(t)$ thus produced, and from the signal to extract an estimate of the temperature T of the blob.*

duration of its descent. That requirement can be expressed

$$(\text{characteristic molecular speed } \alpha)\cdot(\text{flight time } \sqrt{2h/g}\,) \ll h \tag{4}$$

which gives $T \ll \frac{1}{4}mgh/k$. If—reasonably—we set $h = 5\,\text{cm}$ and assume the vapor to be composed of ^{87}Rb atoms ($m = 1.45\times10^{-25}$ kg) then we obtain $T \ll 10^{-3}$K. Hannah reports that "a well-regulated MOT should trap atoms ranging from 40×10^{-6}K to 2×10^{-3}K," while Wieman *et al*[1] claim that 1μK should be achievable. Those temperatures conform to the TOF condition (4), yet are so much higher than the $T_c \approx 10^{-10}$K that announces the onset of quantum effects that in MOT applications we can expect to enjoy success with a <u>TOF signal analysis based upon classical mechanics & Maxwellian gas theory</u>.[7]

3. Signal produced by a single-speed point source. The figures on the facing page are self-explanatory, and serve in themselves to capture the ballistic and geometrical essence of the argument. Suppose that, by action of some isotropic process, N points/atoms have been sprayed with statistical uniformity onto the surface of the sphere shown in Figure 5. Assume, moreover, that—as suggested by Figures 2 & 4—z and r are time-dependent. The expected number of atoms on the sub-planar cap—the number of atoms that, riding on the sphere, have been transported past the detection plane and "completed their race"—can be described

$$\begin{aligned} F(z,r) &= N\cdot\frac{\text{area of sub-planar cap}}{\text{area of entire sphere}} \\ &= \begin{cases} N\cdot\dfrac{r-z}{2r} & \text{if } z^2\leqslant r^2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and the *rate* at which atoms drift through the detection plane—physically: the "signal"—becomes

$$\begin{aligned} S(t) = \tfrac{d}{dt}F\big(z(t),r(t)\big) &= \begin{cases} N\cdot\dfrac{z\dot r - r\dot z}{2r^2} & \text{if } z^2\leqslant r^2 \\ 0 & \text{otherwise} \end{cases} \\ &= N\Big\{\theta(z+r)-\theta(z-r)\Big\}\cdot\frac{z\dot r - r\dot z}{2r^2} \end{aligned} \tag{5}$$

We have special interest in the case

$$\left.\begin{aligned} z(t) &= h + ut - \tfrac{1}{2}gt^2 \\ r(t) &= vt \end{aligned}\right\} \tag{6}$$

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[7] We remark in this connection that at $T_c \approx 10^{-10}$K a rubidium atom can be expected to be moving at about

$$v_c = \sqrt{2kT_c/m} \approx 0.14 \text{ mm/s}$$

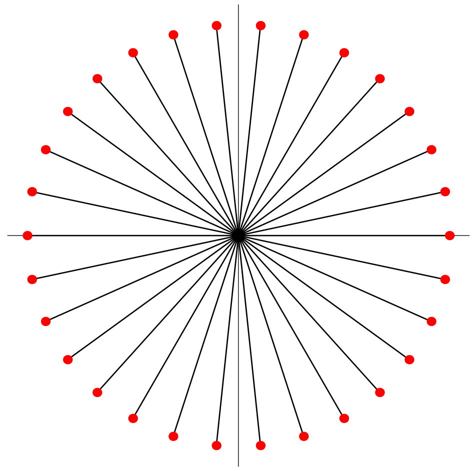

FIGURE 3: *Identical particles have fled isotropically with identical speeds v.*

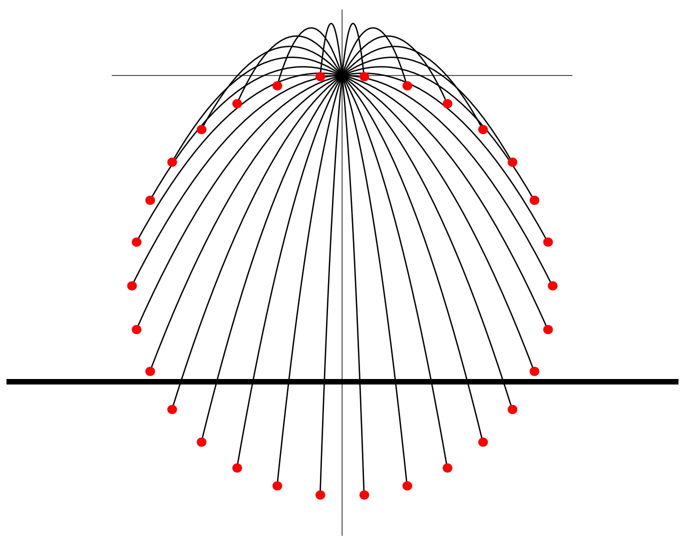

FIGURE 4: *"Fireworks display." Same physics as above, but in the presence of a gravitational field (or equivalently: as viewed by an upwardly accelerated observer). We have interest in the rate at which debris passes through the "detection plane" (represented in the figure by the heavy horizontal line) and realized physically by a slab-like laser beam.*

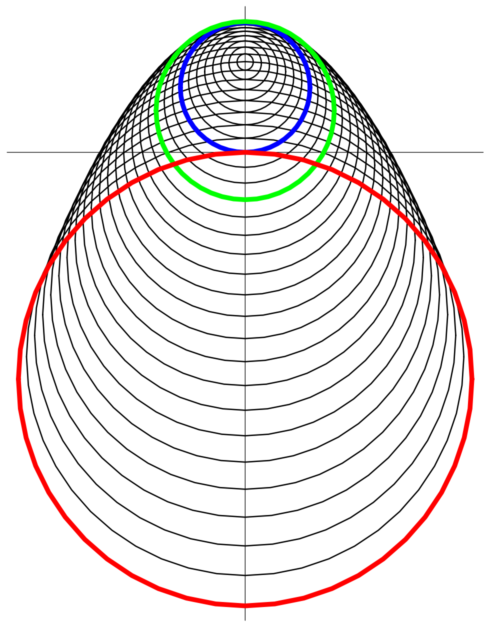

FIGURE 5: *Stroboscopic representation of a uniformly expanding dropped sphere. The blue sphere is making first contact with the detection plane, the red sphere is making final contact. The TOF signal is proportional to the rate at which the sub-planar area is increasing.*

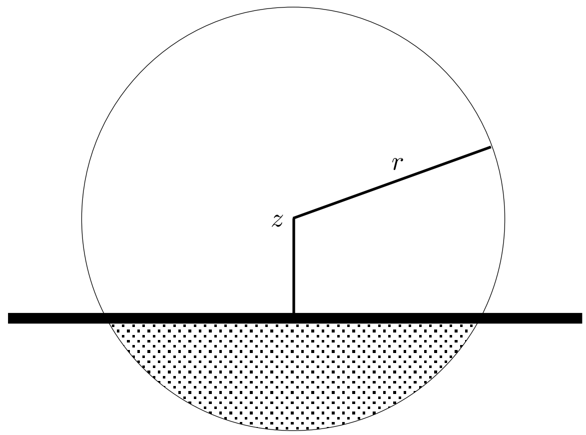

FIGURE 6: *A little elementary calculus serves to establish that the area of the spherical cap below the bold line (which represents the detection plane) can be described*

$$A(z,r) = 2\pi r^2\big[1 - \tfrac{z}{r}\big] \quad \text{which becomes} \quad \begin{cases} 0 & \text{at} \quad z = +r \\ 2\pi r^2 & \text{at} \quad z = 0 \\ 4\pi r^2 & \text{at} \quad z = -r \end{cases}$$

—the essential presumption being that $r \leqslant z \leqslant -r$; i.e., that $z^2 \leqslant r^2$. From the fact that A depends linearly upon z it follows—somewhat counterintuitively—that slices of equal thickness, wherever they may be taken from a sphere, all have the same surface area.

with $v > 0$. We will say that the population was "*dropped* from height h" if $u = 0$, and in the contrary case that it was tossed or "lofted."[8] Returning with (6) to (5) we obtain

$$\begin{aligned} S(t) &= N\Big\{\theta(h + ut - \tfrac{1}{2}gt^2 + vt) - \theta(h + ut - \tfrac{1}{2}gt^2 - vt)\Big\} \\ &\qquad\qquad \cdot \frac{(h + ut - \frac{1}{2}gt^2)v - vt(u - gt)}{2(vt)^2} \\ &= N\Big\{\theta(h + ut - \tfrac{1}{2}gt^2 + vt) - \theta(h + ut - \tfrac{1}{2}gt^2 - vt)\Big\} \cdot \frac{h + \frac{1}{2}gt^2}{2vt^2} \end{aligned} \tag{7}$$

A simple argument establishes that the "switch factor" $\big\{etc.\big\}$ snaps on at the moment the expanding/falling sphere first strikes the detection plane

[8] Experimentalists inform me of their suspicion that lofting may be a fact of life, an uncontrolled side-effect of the abrupt de-confinement of trapped atoms. We want to be in position to estimate the magnitude of the error thus introduced into their measurements.

$$t_{\text{first}} = \frac{\sqrt{(u-v)^2 + 2gh} + (u-v)}{g} \tag{8.1}$$

and snaps off the instant

$$t_{\text{last}} \;= \frac{\sqrt{(u+v)^2 + 2gh} + (u+v)}{g} \tag{8.2}$$

the sphere sinks below the plane. Equation (7) can in this notation be rendered

$$S(t) = N\Big\{\theta(t - t_{\text{first}}) - \theta(t - t_{\text{last}})\Big\} \cdot \frac{h + \frac{1}{2}gt^2}{2vt^2} \tag{8.3}$$

and we are gratified to discover that *Mathematica*, working from (8), supplies

$$\int_{t_{\text{first}}}^{t_{\text{last}}} S(t)\,dt = N \quad : \quad \text{all parameter assignments}$$

4. Signal produced by a Maxwellian point source. Bringing (1) to (7) we obtain the signal

$$\begin{aligned} &S_{\text{thermalized point}}(t) \\ &\quad = N \cdot \int_0^\infty \Big(\frac{1}{\alpha\sqrt{\pi}}\Big)^3 e^{-(v/\alpha)^2} 4\pi v \Big\{\theta(vt+z) - \theta(-vt+z)\Big\} \cdot \frac{h + \frac{1}{2}gt^2}{2t^2}\,dv \\ &\qquad\qquad z \equiv h + ut - \tfrac{1}{2}gt^2 \end{aligned} \tag{9}$$

of a *Maxwellian superposition* of such populations. The integration problem can be approached in several ways, the most straightforward of which was brought to my attention by David Griffiths, who observed that because

$$\theta(vt+z) - \theta(-vt+z) = \begin{cases} -1 & : \quad v < -z/t \quad \text{(unphysical)} \\ \;\;0 & : \quad -z/t < v < +z/t \\ +1 & : \quad v > +z/t \end{cases}$$

we have

$$S_{\text{thermalized point}}(t) = N \cdot \Big(\frac{1}{\alpha\sqrt{\pi}}\Big)^3 4\pi \,\frac{h + \frac{1}{2}gt^2}{2t^2} \int_{z/t}^\infty v\,e^{-(v/\alpha)^2}\,dv$$

The integral is elementary, and immediately gives

$$= N \cdot \frac{h + \frac{1}{2}gt^2}{\alpha t^2\sqrt{\pi}} \cdot \exp\Big\{-\frac{(h + ut - \frac{1}{2}gt^2)^2}{\alpha^2 t^2}\Big\} \tag{10}$$

which in the case $u = 0$ is precisely the result obtained (in quite another way) by Yavin *et al*, and that is fundamental to their paper. . . as it is also to most of what follows. Figures 7 and 8 illustrate the signals produced by dropped/lofted Maxwellian point sources in some representative cases.

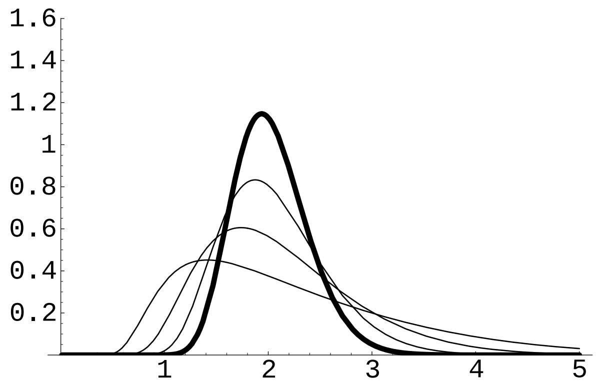

FIGURE 7: **Dropped Maxwellian point source.** *Curves derived from (10) in the case* $u = 0$*. In all cases* $g = 1$ *and* $h = 2$*. The heavy curve arose from* $\alpha = 0.5$*; progressively broader curves arose from setting* $\alpha = 0.7, 1.0, 1.5$*; i.e., from increasing the temperature.*

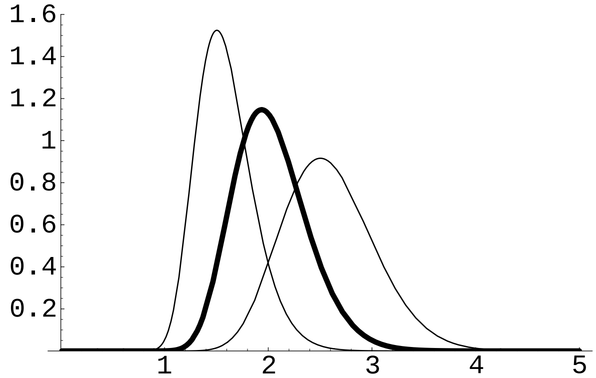

FIGURE 8: **Lofted Maxwellian point source.** *Lofting at speeds* u *much greater than about*

$$\frac{\text{characteristic diameter of workspace}}{\text{time } \sqrt{2h/g} \text{ required to fall a distance } h}$$

would very likely toss the sample right out of the apparatus. In all of the cases plotted $\alpha = 0.5$, $g = 1$, $h = 2$ *(and therefore* $\sqrt{2h/g} = 2$*). The heavy curve arose (as in the preceding figure) from setting* $u = 0$*. The broadened late-arriving curve arose from up-lofting the point source with speed* $u = 0.5$*, the narrowed early-arriving curve arose from down-lofting with that same speed.*

5. Signal produced by a Maxwellian blob. Gas balls can be considered to be "point-like" only in the approximation that

$$\frac{\text{ball diameter}}{\text{mean distance to detection plane}} \ll 1$$

To study departures from that idealized approximation one must take into account the fact that different parts of the initial gas ball (or "blob") must fall different distances before they are detected. This is easily done, at least in simple cases: one has only to proceed from the conjectured design of the blob to a "height density function"

$$n(h)\,dh \equiv \text{number of atoms in the neighborhood } dh \text{ of height } h$$

and to construct integrals of the form

$$S_{\text{thermalized blob}}(t) = \int S(t;\alpha,g,h,u)\,n(h)\,dh$$

This I have done[6] in the illustrative cases of a **spherically symmetric Gaussian blob**

$$n(h) = N\frac{1}{R\sqrt{2\pi}}\exp\Big\{-\frac{(h-h_0)^2}{2R^2}\Big\}$$

and a **uniformly dense spherical blob**

$$n(h) = \begin{cases} N\dfrac{3}{4R^3}\big[R^2-(h-h_0)^2\big] & : \quad h_0 - R \leqslant h \leqslant h_0 + R \\ 0 & : \quad \text{otherwise} \end{cases}$$

The results are shown in the following figures:

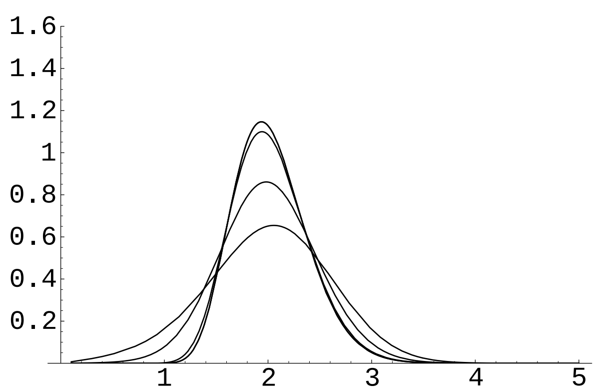

FIGURE 9: **Dropped Gaussian blob.** *Here superimposed are the signals that result from setting R to $0\%, 1\%, 10\%, 30\%$ and 50% of h_0. The parameters α, g and h_0 have been assigned the same values as in Figure 7. At the 1% setting the extended source effect is imperceptible.*

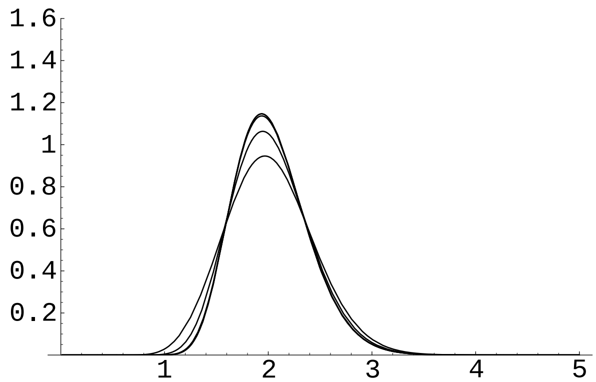

FIGURE 10: **Dropped uniformly dense spherical blob.** *The R/h_0 ratios and parameter setting are identical to those in the preceding figure. The extended source effect is even less pronounced.*

The short of it: blobbiness fuzzes the signal, but the effect is slight in realistic cases.

6. New method for extracting temperature from signal data. An atom thrust downward with speed $\alpha \equiv \sqrt{2kT/m}$ from height $h - R$ will intercept the detection plane at time

$$\begin{aligned} t_- &= \frac{\sqrt{\alpha^2 + 2g(h-R)} - \alpha}{g} \\ &= \frac{\sqrt{\alpha^2 + 2gh} - \alpha}{g} - \frac{1}{\sqrt{\alpha^2 + 2gh}}R - \frac{g}{2(\alpha^2 + 2gh)^{\frac{3}{2}}}R^2 - \cdots \end{aligned}$$

while an atom tossed upward with that same speed from height $h + R$ will be detected at time

$$\begin{aligned} t_+ &= \frac{\sqrt{\alpha^2 + 2g(h+R)} + \alpha}{g} \\ &= \frac{\sqrt{\alpha^2 + 2gh} + \alpha}{g} + \frac{1}{\sqrt{\alpha^2 + 2gh}}R - \frac{g}{2(\alpha^2 + 2gh)^{\frac{3}{2}}}R^2 + \cdots \end{aligned}$$

Therefore

$$\begin{aligned} t_+ - t_- &= \frac{2\alpha}{g} + \frac{2}{\sqrt{2gh + \alpha^2}}R + \cdots \\ &= R\frac{1}{\sqrt{\frac{1}{2}gh}} + \frac{2\alpha}{g} + R\frac{1}{(2gh)^{\frac{3}{2}}}\alpha^2 + \cdots \end{aligned}$$

and in the point-source approximation ($R = 0$) we have the simple relation

$$\alpha = \tfrac{1}{2}g(t_+ - t_-) \tag{11}$$

It is with the aid of (11) that experimentalists standardly attempt to extract temperature estimates from signal data. The problem is that trapped populations contain (at thermal equilibrium) atoms of *assorted* velocities, so the resulting signal cannot have a clearly defined "beginning" and "end." I turn now to the description of a practical alternative to (11) which was suggested by Figure 7. Introduce

$$\Omega \equiv \frac{\text{curvature at the signal max}}{\text{maximal signal value}}$$

and by straightforward computation obtain

$$\Omega(t_{\max};\alpha,g,h,0) = -\frac{2g^2}{\alpha^2} - \frac{g}{2h} - \frac{\alpha^2}{2h^2} - \cdots$$

which in leading approximation (low temperatures $\Leftrightarrow$ α small) becomes

$$\alpha^2 \equiv 2kT/m = -\frac{2g^2}{\Omega} \tag{12}$$

The point is that Ω can be extracted from data even in situations where the meaning of $t_+ - t_-$ has become vague.

7. Error introduced by soft lofting. Retain the convention

$$u \equiv \text{speed with which gas ball was lofted}$$

and let

$$v_0 \equiv \sqrt{2gh} = \begin{cases}\text{speed with which a gas ball dropped from}\\ \text{height } h \text{ passes through the detection plane}\end{cases}$$

"Soft lofting" refers to the circumstance

$$u/v_0 \ll 1$$

Taylor expansion of results already in hand leads to these leading-order refinements of (11) and (12):

$$\alpha^2 = \tfrac{1}{4}g^2(t_+ - t_-)^2\cdot\big[1 - 2u/v_0\big] \tag{11*}$$

$$\alpha^2 = -\frac{2g^2}{\Omega}\cdot\big[1 - 2u/v_0\big] \tag{12*}$$

Evidently the Ω-method and the $(t_+ - t_-)$-method are *equally/identically sensitive to soft-lofting*. Up-lofting tends, however, to increase the ambiguity that attaches in all cases to $(t_+ - t_-)$, and this is a little problem to which the Ω-method is immune.

8. Interesting mathematical detail: a toy perturbation theory. The argument that leads to (12*) is conceptually/computationally straightforward, but hinges on a mathematical point that I had not encountered before. Let $p(x)$ and $q(x)$ be Taylor-expandable about x_0 and suppose, more particularly, that $p(x_0) = 0$. Construct

$$P(x) \equiv p(x) + \epsilon\, q(x)$$

and require that $X \equiv x_0 + x_1\epsilon + x_2\epsilon^2 + x_3\epsilon^3 + \cdots$ be a root of $P(x)$. With the assistance of *Mathematica* we are led quickly to the conclusion that necessarily

$$x_1 = -\frac{q(x_0)}{p'(x_0)}$$

$$x_2 = -\frac{q(x_0)[q(x_0)\,p''(x_0) - 2q'(x_0)\,p'(x_0)]}{2[p'(x_0)]^3}$$

which we abbreviate

$$= -\frac{q}{2[p']^3}\Big\{q\,p'' - 2q'p'\Big\}$$

and that in next higher order

$$x_3 = -\frac{q}{6[p']^5}\Big\{6[p']^2[q']^2 - 9qp'q'p'' + 3q^2[p'']^2 + 3q[p']^2q'' - q^2p'p'''\Big\}$$

$$\vdots$$

It is clear on a moment's thought why $x_1 = x_2 = x_3 = \cdots = 0$ if $q(x_0) = 0$, and is interesting to consider why the method fails when $p'(x_0) = 0$, but discussion of that matter would take me rapidly too far afield. In a companion essay I explore those details, and indicate what this "toy perturbation theory" has to do with real (quantum mechanical) perturbation theory.